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ORIGINAL PAPER

Stationary patterns in a two-cell coupled isothermal chemical system with arbitrary powers of autocatalysis

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Abstract Stationary patterns of a two-cell coupled isothermal chemical system with arbitrary powers of autocatalysis are considered. Firstly, we obtain the stability of the unique positive constant equilibrium solution for the system. Then, based on a priori estimates, non-existence and existence of nontrivial steady state solutions are shown by using implicit function theorem and topological degree theory, respectively. The effects of autocatalysis order and diffusion coefficients to the pattern formation are discussed.

Keywords Autocatalytic chemical system · Arbitrary powers of autocatalysis · Leray–Schauder degree · Pattern

Mathematics Subject Classification (2010) 35K57 · 35K55

1 Introduction

In the spatially inhomogeneous case, a lot of coupled partial differential equations have been proposed by chemists, biologists and mathematicians to model problems arising from various subjects such as chemical reactions, genetics and population dynamics.

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The existence of non-constant time-independent positive solutions, also called stationary patterns, is an indication of the richness of the corresponding partial differential equation dynamics.

In 1952, Turing published a paper “The chemical basis of morphogenesis” [1], which is now regarded as the foundation of basic chemical theory or reaction–diffusion theory of morphogenesis. Turing suggested that, under certain conditions, chemicals can react and diffuse in such a way as to produce non-constant equilibrium solutions, which represent spatial patterns of chemical or morphogenesis concentration. So far, a variety of patterns have been observed and studied in disparate systems. These include chemical models such as the activator-inhibitor Gierer–Meinhardt model [2,3], the Brusselator model [4,5], the Sel'kov model [6,7], the Gray–Scott model [8,9], the Noyes-Field model for Belousov-Zhabotinskii reaction [10], and the biological models include the competition model [11,12] and the predator-prey model [13–18].

In biochemical procedures autocatalytic chemical reactions have been identified as one of main nonlinear mechanisms [19–21]. A simplest autocatalytic chemical reaction is of the following form:

$$A + mB \longrightarrow (m+1)B,$$

where A is the reactant and B is the autocatalyst, and m is the order of the autocatalysis. Previous studies mainly concentrated on the situation when m is a positive integer, in particular, (i) $m = 1$, i.e., quadratic autocatalysis [22–24]; (ii) $m = 2$, cubic autocatalysis [25–28]. However, in general, the order m in realistic models is determined empirically, and so m is not necessarily an integer (see Kay et al. [29]). Here we discuss the general case $m > 0$. We consider the prototype chemical reaction scheme based on arbitrary order autocatalysis

$$P \xrightarrow{k_0} A, \quad A + mB \xrightarrow{k_1} (m+1)B, \quad B \xrightarrow{k_2} C, \quad A \xrightarrow{k_3} B,$$

in which P, A, B and C are certain chemical species with molar concentrations p, a, b and c, respectively, and k_i $(i = 0, 1, 2, 3)$ are constants representing the reaction rates. We assume that the concentration of P and C are independent of time, and the concentration of P remains constant at its initial value p_0. Moreover, it is assumed that the cells are sufficiently thin so that transverse diffusion across them can be considered to be instantaneous (see [25]). So we have two identical regions, divided by a semipermeable membrane which allows the passage of autocatalyst B only, with some reaction taking place in each region. The equations are obtained as in [25], but now with the arbitrary autocatalysis order included, namely

$$\begin{cases} \dfrac{da_1}{d\bar{t}} = k_0 p_0 - k_1 a_1 b_1^m - k_3 a_1, \\ \dfrac{db_1}{d\bar{t}} = k_1 a_1 b_1^m + k_3 a_1 - k_2 b_1 + \beta(b_2 - b_1), \\ \dfrac{da_2}{d\bar{t}} = k_0 p_0 - k_1 a_2 b_2^m - k_3 a_2, \\ \dfrac{db_2}{d\bar{t}} = k_1 a_2 b_2^m + k_3 a_2 - k_2 b_2 + \beta(b_1 - b_2), \end{cases} \tag{1.1}$$

where (a_1, b_1) and (a_2, b_2) are the concentrations in cells 1 and 2, respectively, β is the dimensionless coupling parameter, and $m > 0$ is the order of the autocatalysis.

For simplicity, we use the following scaling to (1.1):

$$u = \left(\frac{k_1}{k_2}\right)^{\frac{1}{m}} a_1, \quad v = \left(\frac{k_1}{k_2}\right)^{\frac{1}{m}} b_1, \quad w = \left(\frac{k_1}{k_2}\right)^{\frac{1}{m}} a_2, \quad z = \left(\frac{k_1}{k_2}\right)^{\frac{1}{m}} b_2, \quad t = k_2 \bar{t}.$$

Then system (1.1) becomes

$$\begin{cases} \dfrac{du}{dt} = a - uv^m - bu, \\ \dfrac{dv}{dt} = uv^m + bu - v + c(z - v), \\ \dfrac{dw}{dt} = a - wz^m - bw, \\ \dfrac{dz}{dt} = wz^m + bw - z + c(v - z), \end{cases}$$

where $a = \left(\frac{k_1}{k_2}\right)^{\frac{1}{m}} \frac{k_0 p_0}{k_2}$, $b = \frac{k_3}{k_2}$, and $c = \frac{\beta}{k_2}$.

To consider the reaction scheme (1.1) taking place in a closed vessel without stirring, we have the following reaction–diffusion system:

$$\begin{cases} u_t - d_1 \Delta u = a - uv^m - bu & \text{in } \Omega \times (0, \infty), \\ v_t - d_2 \Delta v = uv^m + bu - v + c(z - v) & \text{in } \Omega \times (0, \infty), \\ w_t - d_1 \Delta w = a - wz^m - bw & \text{in } \Omega \times (0, \infty), \\ z_t - d_2 \Delta z = wz^m + bw - z + c(v - z) & \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_\nu z = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x) & \text{on } \overline{\Omega}, \\ w(x, 0) = w_0(x), \ z(x, 0) = z_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{1.2}$$

where d_1 and d_2 are the diffusion coefficients of reactant A and autocatalyst B, respectively. The admissible initial data $u_0(x)$, $v_0(x)$, $w_0(x)$ and $z_0(x)$ are smooth nonnegative functions which are not identically zero.

In the present paper, our main interest is in the stationary patterns generated by the above reaction–diffusion system. This leads us to investigate the associated steady-state problem, which satisfies the following coupled elliptic system:

$$\begin{cases} -d_1\Delta u = a - uv^m - bu & \text{in } \Omega, \\ -d_2\Delta v = uv^m + bu - v + c(z-v) & \text{in } \Omega, \\ -d_1\Delta w = a - wz^m - bw & \text{in } \Omega, \\ -d_2\Delta z = wz^m + bw - z + c(v-z) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_\nu z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here, $\Omega \subset \mathbf{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, a, b, c, m, d_1, d_2 are positive constants, ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \frac{\partial}{\partial\nu}$. The homogeneous Neumann boundary condition means that system is self-contained with zero flux across the boundary.

In a series of papers, we have looked at the stationary patterns that arise in a simple prototype chemical reaction due to chemical coupling. In one spatial dimension, Leach and Wei [30] considered pattern formation in a coupled system of reaction–diffusion equations. Hubbard et al. [31] extend their preliminary work of [30] by considering the spatial dimension $N = 2$. Ghergu [5] studied the non-constant steady states for Brusselator type systems in any spatial dimension. For the more information about pattern formation in coupled chemical systems, one may refer to [8,32–35], etc. In the case of two-cell coupled chemical systems, Hill and Merkin [26] considered a coupled system whereby two identical cells are coupled via the diffusive interchange of the autocatalyst in one spatial dimension. Zhang and Liu [36] discussed the spatiotemporal structures arising in two identical cells, each governed by arbitrary order autocatalator kinetics and coupled via the diffusive interchange of a reactant. In [37,38], You and Zhou proved the existence of a global attractor for the solution semiflow of the coupled two-cell Brusselator and extended Brusselator systems. Zhou and Mu [39] studied the stationary problem for the coupled two-cell Brusselator model [37]. Based on the works of Ghergu [5] and Zhou and Mu [39], in [40], the first author of this work investigated the pattern formation of a general coupled two-cell Brusselator-type system in any spatial dimension.

In this paper, we attempt to further understand the influences of diffusion on pattern formation in the two-cell coupled system (1.3) for any dimension $N \geq 1$. Moreover, the crucial role played by the autocatalysis order m in generating stationary patterns is exhibited, concretely, if $0 < m \leq 1$ then no stationary patterns occur, while if $m > 1$ then may exist such patterns. Our mathematical approach is based on a priori estimates, topological degree theory and implicit function theorem.

Throughout this paper, the positive solution (u, v, w, z) satisfying (1.3) refers to a classical one, i.e., $(u, v, w, z) \in [C^2(\overline{\Omega})]^4$ such that u, v, w, z are positive on $\overline{\Omega}$. Simple computation shows that

$$(u^*, v^*, w^*, z^*) = \left(\frac{a}{a^m + b}, a, \frac{a}{a^m + b}, a\right)$$

is the unique constant solution of (1.3) and it is clear that only nonnegative solutions of (1.3) are of real interest.

The remaining part of this paper is organized as follows. In Sect. 2, we discuss the stability of the unique constant steady state solution of (1.2). In Sect. 3, we establish a priori estimates for nonnegative solutions of (1.3). In Sect. 4, we consider the non-existence of non-constant positive steady state solutions of (1.3), while Sect. 5 is devoted to the existence of non-constant positive solutions to (1.3). Finally, we give some discussion of our main results in Sect. 6.

2 Stability analysis

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Set $\mathbf{X}_j$ is the eigenspace corresponding to μ_j. Let

$$\mathbf{X} = \left\{(u, v, w, z) \in \left[C^2(\overline{\Omega})\right]^4 \mid \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_\nu z = 0 \quad \text{on } \partial\Omega\right\},$$

$\{\phi_{jl}; l = 1, \ldots, m(\mu_j)\}$ be an orthonormal basis of $\mathbf{X}_j$, and $\mathbf{X}_{jl} = \{\mathbf{c}\phi_{jl} | \ \mathbf{c} \in \mathbf{R}^4\}$. Here $m(\mu_j)$ is the multiplicity of μ_j. Then

$$\mathbf{X} = \bigoplus_{j=0}^{\infty} \mathbf{X}_j \quad \text{and} \quad \mathbf{X}_j = \bigoplus_{l=1}^{m(\mu_j)} \mathbf{X}_{jl}.$$

We note that (1.2) has a unique nonnegative global solution (u, v, w, z) by standard theory of parabolic equations. The aim of this section is to prove the stability of (u^*, v^*, w^*, z^*) to system (1.2).

Theorem 2.1 *The positive equilibrium* (u^*, v^*, w^*, z^*) *to system* (1.2) *is uniformly asymptotically stable provided that*

$$ma^m < (a^m + b)(a^m + b + 1) \tag{2.1}$$

and

$$\mu_1 > \frac{1}{d_2}\frac{(m-1)a^m - b}{a^m + b} - \frac{1}{d_1}(a^m + b). \tag{2.2}$$

Proof The linearization of (1.2) at (u^*, v^*, w^*, z^*) is

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} + \begin{pmatrix} g_1(u - u^*, v - v^*, w - w^*, z - z^*) \\ g_2(u - u^*, v - v^*, w - w^*, z - z^*) \\ g_3(u - u^*, v - v^*, w - w^*, z - z^*) \\ g_4(u - u^*, v - v^*, w - w^*, z - z^*) \end{pmatrix},$$

where $g_i(y_1, y_2, y_3, y_4) = O(y_1^2 + y_2^2 + y_3^2 + y_4^2), i = 1, 2, 3, 4$, and

$$\mathcal{L} = \begin{pmatrix} d_1\Delta - a^m - b & \frac{-ma^m}{a^m+b} & 0 & 0 \\ a^m + b & d_2\Delta + \frac{ma^m}{a^m+b} - c - 1 & 0 & c \\ 0 & 0 & d_1\Delta - a^m - b & \frac{-ma^m}{a^m+b} \\ 0 & c & a^m + b & d_2\Delta + \frac{ma^m}{a^m+b} - c - 1 \end{pmatrix}.$$

For each $j\,(j = 0, 1, 2, \ldots)$, X_j is invariant under the operator $\mathcal{L}$, and ξ_j is an eigenvalue of $\mathcal{L}$ on X_j if and only if ξ_j is an eigenvalue of the following matrix

$$A_j = \begin{pmatrix} -d_1\mu_j - a^m - b & \frac{-ma^m}{a^m+b} & 0 & 0 \\ a^m + b & -d_2\mu_j + \frac{ma^m}{a^m+b} - c - 1 & 0 & c \\ 0 & 0 & -d_1\mu_j - a^m - b & \frac{-ma^m}{a^m+b} \\ 0 & c & a^m + b & -d_2\mu_j + \frac{ma^m}{a^m+b} - c - 1 \end{pmatrix},$$

i.e., ξ_j satisfies the following equation

$$\left[(-d_1\mu_j - a^m - b - \xi_j)\left(-d_2\mu_j + \frac{ma^m}{a^m+b} - 2c - 1 - \xi_j\right) + ma^m\right]$$
$$\times\left[(-d_1\mu_j - a^m - b - \xi_j)\left(-d_2\mu_j + \frac{ma^m}{a^m+b} - 1 - \xi_j\right) + ma^m\right] = 0.$$

Denote

$$B_j^{(1)} = \begin{pmatrix} -d_1\mu_j - a^m - b & \frac{-ma^m}{a^m+b} \\ a^m + b & -d_2\mu_j + \frac{ma^m}{a^m+b} - 1 \end{pmatrix},$$
$$B_j^{(2)} = \begin{pmatrix} -d_1\mu_j - a^m - b & \frac{-ma^m}{a^m+b} \\ a^m + b & -d_2\mu_j + \frac{ma^m}{a^m+b} - 2c - 1 \end{pmatrix}.$$

From the above analysis, we know that ξ_j is an eigenvalue of $\mathcal{L}$ if and only if ξ_j is an eigenvalue of $B_j^{(1)}$ or $B_j^{(2)}$. So, in order to analyze the eigenvalue of $\mathcal{L}$, it is sufficient to analyze the eigenvalue of $B_j^{(1)}$ and $B_j^{(2)}$.

We first consider the matrix $B_j^{(1)}$. The direct calculation gives

$$\det B_j^{(1)} = \mu_j\left[d_1 d_2\mu_j + d_2(a^m + b) + d_1\left(1 - \frac{ma^m}{a^m+b}\right)\right] + (a^m + b),$$
$$\mathrm{Tr}B_j^{(1)} = -(d_1 + d_2)\mu_j + \frac{ma^m}{a^m+b} - (a^m + b + 1),$$

where $\det B_j^{(1)}$ and $\mathrm{Tr}B_j^{(1)}$ are respectively the determinant and trace of $B_j^{(1)}$. Note that $\mu_0 = 0$, and so it is easy to check that $\det B_j^{(1)} > 0$ and $\mathrm{Tr}B_j^{(1)} < 0$ under the conditions of Theorem 2.1.

As above, we then consider the matrix $B_j^{(2)}$. By virtue of

$$\det B_j^{(2)} = \mu_j\left[d_1 d_2 \mu_j + d_2(a^m + b) + d_1\left(1 + 2c - \frac{ma^m}{a^m + b}\right)\right] + (a^m + b),$$
$$\mathrm{Tr} B_j^{(2)} = -(d_1 + d_2)\mu_j + \frac{ma^m}{a^m + b} - (a^m + b + 2c + 1),$$

we can easily check that $\det B_j^{(2)} > 0$ and $\mathrm{Tr} B_j^{(2)} < 0$.

A similar argument to [7] gives that the spectrum of $\mathcal{L}$ lies in $\{\mathrm{Re}\ \xi < -\delta\}$ for some positive δ independent of $j \geq 0$. It is known that (u^*, v^*, w^*, z^*) is uniformly asymptotically stable and the proof is complete. □

If $0 < m \leq 1$, then conditions (2.1) and (2.2) are satisfied. In this case, we have

Corollary 2.1 *If $0 < m \leq 1$ holds, then (u^*, v^*, w^*, z^*) to system (1.2) is uniformly asymptotically stable.*

Remark 2.1 From Theorem 2.1 and Corollary 2.1, we see that the positive constant solution (u^*, v^*, w^*, z^*) to system (1.2) is uniformly asymptotically stable if (i) $0 < m \leq 1$, or (ii) $m > 1$, a, b, m satisfying (2.1) be fixed, and d_1 is small or d_2 is large such that (2.2) holds. As a result, in the above two cases, it is impossible to expect the bifurcation of (1.3) near (u^*, v^*, w^*, z^*), and it seems difficult to capture the patterns of (1.3). In turn, when $m > 1$, d_1 is large or d_2 is small, such that (2.2) dose not hold, then system (1.3) may have non-constant positive solutions, as will be seen in Sect. 5.

3 A priori estimates

In this section, we are ready to derive a priori upper and lower bounds for all positive solutions to (1.3). To this end, we first cite a known result.

Lemma 3.1 (Maximum principle [41]) *Suppose that $g \in C(\overline{\Omega} \times \mathbf{R})$.*

(i) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \ \text{ in } \Omega, \quad \partial_\nu w \leq 0 \ \text{ on } \partial\Omega.$$

If $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.

(ii) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \ \text{ in } \Omega, \quad \partial_\nu w \geq 0 \ \text{ on } \partial\Omega.$$

If $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.

Theorem 3.1 *Every positive solution* (u, v, w, z) *of (1.3) satisfies*

$$\frac{a}{b+2^m a^m(1+\frac{d_1}{d_2 b})^m} \le u(x) \le \frac{a}{b}, \tag{3.1}$$

$$\frac{ab}{(c+1)\Big[b+2^m a^m(1+\frac{d_1}{d_2 b})^m\Big]} \le v(x) \le 2a\Big(1+\frac{d_1}{d_2 b}\Big), \tag{3.2}$$

$$\frac{a}{b+2^m a^m(1+\frac{d_1}{d_2 b})^m} \le w(x) \le \frac{a}{b}, \tag{3.3}$$

and

$$\frac{ab}{(c+1)\Big[b+2^m a^m(1+\frac{d_1}{d_2 b})^m\Big]} \le z(x) \le 2a\left(1+\frac{d_1}{d_2 b}\right). \tag{3.4}$$

Proof Suppose that (u, v, w, z) is a positive solution of (1.3). We set

$$u(x_1) = \max_{\overline{\Omega}} u, \quad v(x_2) = \max_{\overline{\Omega}} v, \quad w(x_3) = \max_{\overline{\Omega}} w, \quad z(x_4) = \max_{\overline{\Omega}} z,$$

and

$$u(y_1) = \min_{\overline{\Omega}} u, \quad v(y_2) = \min_{\overline{\Omega}} v, \quad w(y_3) = \min_{\overline{\Omega}} w, \quad z(y_4) = \min_{\overline{\Omega}} z.$$

Applying Lemma 3.1 to the first equation in (1.3), we obtain that

$$a - u(x_1)v^m(x_1) - bu(x_1) \ge 0,$$

which implies

$$u(x_1) \le \frac{1}{b}\Big[a - u(x_1)v^m(x_1)\Big] \le \frac{a}{b}. \tag{3.5}$$

Using the same argument as above, we have

$$a - w(x_3)z^m(x_3) - bw(x_3) \ge 0,$$

which yields

$$w(x_3) \le \frac{1}{b}\left(a - w(x_3)z^m(x_3)\right) \le \frac{a}{b}. \tag{3.6}$$

Set $\phi = d_1(u+w) + d_2(v+z)$. Adding the first four relations in (1.3), one gets

$$-\Delta\phi = 2a - (v+z) \ \text{ in } \ \Omega, \quad \partial_\nu \phi = 0 \ \text{ on } \ \partial\Omega.$$

Let now $x_5 \in \overline{\Omega}$ be a maximum point of ϕ. According to Lemma 3.1 we have

$$2a - (v(x_5) + z(x_5)) \geq 0,$$

that is, $v(x_5) + z(x_5) \leq 2a$. By virtue of (3.5) and (3.6), it follows that

$$\begin{aligned} d_2 v(x_2) \leq \phi(x_2) \leq \phi(x_5) &= d_1(u(x_5) + w(x_5)) + d_2(v(x_5) + z(x_5)) \\ &\leq d_1(u(x_1) + w(x_3)) + 2ad_2 \leq 2a\left(\frac{d_1}{b} + d_2\right). \end{aligned}$$

This yields

$$v(x_2) \leq 2a\left(1 + \frac{d_1}{bd_2}\right). \tag{3.7}$$

Arguing in a similar way, we can obtain that

$$\begin{aligned} d_2 z(x_4) \leq \phi(x_4) \leq \phi(x_5) &= d_1(u(x_5) + w(x_5)) + d_2(v(x_5) + z(x_5)) \\ &\leq d_1(u(x_1) + w(x_3)) + 2ad_2 \leq 2a\left(\frac{d_1}{b} + d_2\right), \end{aligned}$$

which implies

$$z(x_4) \leq 2a\left(1 + \frac{d_1}{bd_2}\right). \tag{3.8}$$

On the other hand, by the first equation of (1.3) and Lemma 3.1, we obtain

$$a - u(y_1)v^m(y_1) - bu(y_1) \leq 0,$$

that is,

$$a \leq u(y_1)\left(b + v^m(y_1)\right),$$

which combined with (3.7) yields

$$u(y_1) \geq \frac{a}{b + v^m(y_1)} \geq \frac{a}{b + 2^m a^m \left(1 + \frac{d_1}{bd_2}\right)^m}. \tag{3.9}$$

Similarly, applying Lemma 3.1 to the third equation in (1.3) yields

$$a - w(y_3)z^m(y_3) - bw(y_3) \leq 0 \Rightarrow a \leq w(y_3)\left(b + z^m(y_3)\right),$$

which implies

$$w(y_3) \geq \frac{a}{b + z^m(y_3)} \geq \frac{a}{b + 2^m a^m \left(1 + \frac{d_1}{bd_2}\right)^m}. \tag{3.10}$$

Next, by the second equation of (1.3) and Lemma 3.1, we have

$$u(y_2)v^m(y_2) + bu(y_2) - v(y_2) + c(z(y_2) - v(y_2)) \leq 0.$$

In view of (3.9), the above inequality becomes

$$\begin{aligned}(c+1)v(y_2) &\geq u(y_2)v^m(y_2) + bu(y_2) + cz(y_2) \geq bu(y_2) \\ &\geq bu(y_1) \geq \frac{ab}{b + 2^m a^m \left(1 + \frac{d_1}{bd_2}\right)^m}.\end{aligned}$$

Therefore, we find that

$$v(y_2) \geq \frac{ab}{(c+1)\left[b + 2^m a^m \left(1 + \frac{d_1}{d_2 b}\right)^m\right]}. \tag{3.11}$$

Arguing in a similar way, Lemma 3.1 applied to the forth equation of (1.3) yields

$$z(y_4) \geq \frac{ab}{(c+1)\left[b + 2^m a^m \left(1 + \frac{d_1}{d_2 b}\right)^m\right]}. \tag{3.12}$$

In view of (3.5)–(3.12) we obtain Theorem 3.1. □

From Theorem 3.1, the following corollary is obvious.

Corollary 3.1 *Let $a, b, c, m, D_1, D_2 > 0$ be fixed. Then, there exist two positive constants $C_1, C_2 > 0$ depending on a, b, c, m, D_1, D_2 such that for all $0 < d_1 \leq D_1$ and $d_2 \geq D_2$, every positive solution (u, v, w, z) of (1.3) satisfies*

$$C_1 < u, v, w, z < C_2 \quad \text{in } \overline{\Omega}.$$

Using the standard results of elliptic regularity and embedding theory (see, e.g., [42]), we can further improve Corollary 3.1 and obtain the following result.

Theorem 3.2 *Let $a, b, c, m, D_1, D_2 > 0$ be fixed. Then, for any positive integer $k \geq 1$ there exists a constant*

$$C = C(a, b, c, m, D_1, D_2, k, N, \Omega) > 0$$

such that for all $0 < d_1 \leq D_1$ and $d_2 \geq D_2$, every positive solution (u, v, w, z) of (1.3) belongs to $[C^\infty(\overline{\Omega})]^4$ and satisfies

$$\| u \|_{C^k(\overline{\Omega})} + \| v \|_{C^k(\overline{\Omega})} + \| w \|_{C^k(\overline{\Omega})} + \| z \|_{C^k(\overline{\Omega})} \leq C.$$

4 Non-existence of non-constant positive solutions

In this section, by using the implicit function theorem, we investigate the non-existence of non-constant positive solutions of (1.3) in two cases: (i) d_2 is large enough; (ii) a is small enough. Our idea comes from [10,43]. First of all, we prove the following result.

Lemma 4.1 (i) *Fix* a, b, c, m *and* d_1. *Let* (u_i, v_i, w_i, z_i) *be the positive solution of* (1.3) *with* $d_2 = d_{2,i}$ *and* $d_{2,i} \to \infty$ *as* $i \to \infty$. *Then* $(u_i, v_i, w_i, z_i) \to (u^*, v^*, w^*, z^*)$ *in* $[C^2(\overline{\Omega})]^4$ *as* $i \to \infty$.

(ii) *Fix* b, c, m, d_1 *and* d_2. *Let* (u_i, v_i, w_i, z_i) *be the positive solution of* (1.3) *with* $a = a_i$ *and* $a_i \to 0$ *as* $i \to \infty$. *Then* $(u_i, v_i, w_i, z_i) \to (0, 0, 0, 0)$ *in* $[C^2(\overline{\Omega})]^4$ *as* $i \to \infty$.

Proof We first prove (i). By Theorem 3.1, the embedding theory and the standard regularity theory of elliptic equations (see [42]), there is a subsequence of (u_i, v_i, w_i, z_i), also labelled by itself, such that $(u_i, v_i, w_i, z_i) \to (u, v, w, z)$ in $[C^2(\overline{\Omega})]^4$ as $i \to \infty$. Moreover, $v \equiv \delta$ and $z \equiv \eta$, where δ and η are positive constants and (u, δ, w, η) solves

$$\begin{cases} -d_1 \Delta u = a - \delta^m u - bu & \text{in } \Omega, \\ \int_\Omega [\delta^m u + bu - \delta + c(\eta - \delta)] \mathrm{d}x = 0, & \\ -d_1 \Delta w = a - \eta^m w - bw & \text{in } \Omega, \\ \int_\Omega [\eta^m w + bw - \eta + c(\delta - \eta)] \mathrm{d}x = 0, & \\ \partial_\nu u = \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

From the first equation of (4.1), we see $u = \frac{a}{\delta^m + b}$. Substituting $u = \frac{a}{\delta^m + b}$ in the second equation of (4.1), we have

$$\int_\Omega [a - \delta + c(\eta - \delta)] \mathrm{d}x = 0. \tag{4.2}$$

Similarly, we get from the third and fourth equations of (4.1) that

$$w = \frac{a}{\eta^m + b}, \quad \int_\Omega [a - \eta + c(\delta - \eta)] \mathrm{d}x = 0. \tag{4.3}$$

Then it follows from (4.2) and the second equality of (4.3) that

$$\int_\Omega [2a - (\delta + \eta)] \mathrm{d}x = 0,$$

i.e., $\delta + \eta = 2a$, substituting $\delta = 2a - \eta$ into the second equality of (4.3) yields

$$\int_\Omega (1 + 2c)(a - \eta) \mathrm{d}x = 0,$$

that is, $\eta = a$. In view of $\delta + \eta = 2a$, we get $\delta = a$. Then we obtain $u = w = \frac{a}{a^m + b}$. This ends the proof of the result (i).

Next, we prove (ii). Similar to the proof of (i), we get $(u_i, v_i, w_i, z_i) \to (u, v, w, z)$ in $[C^2(\overline{\Omega})]^4$as $i \to \infty$ and (u, v, w, z) satisfies

$$\begin{cases} -d_1 \Delta u = -uv^m - bu & \text{in } \Omega, \\ -d_2 \Delta v = uv^m + bu - v + c(z - v) & \text{in } \Omega, \\ -d_1 \Delta w = -wz^m - bw & \text{in } \Omega, \\ -d_2 \Delta z = wz^m + bw - z + c(v - z) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = \partial_\nu z = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

It is clear that system (4.4) is the special case of system (1.3) when $a = 0$. Therefore, the estimates (3.1)–(3.4) in Theorem 3.1 imply that $u = v = w = z \equiv 0$. The proof of (ii) is complete and we get Lemma 4.1. □

Now, we state our main result in this section.

Theorem 4.1 (i) *Let $a, b, c, m, d_1 > 0$ be fixed. There exists $D > 0$, which depends only on a, b, c, m, d_1 and Ω, such that (1.3) has no non-constant solution for all $d_2 > D$.*

(ii) *Let $b, c, m, d_1, d_2 > 0$ be fixed. There exists $A > 0$, which depends only on b, c, m, d_1, d_2 and Ω, such that (1.3) has no non-constant solution for all $0 < a < A$.*

Proof First, we prove (i). Define $W^{2,2}_\nu(\Omega) = \{g \in W^{2,2}(\Omega) |\ \partial_\nu g = 0 \text{ on } \partial\Omega\}$ and $W^{2,2}_{\nu,0}(\Omega) = W^{2,2}_\nu(\Omega) \cap L^2_0(\Omega)$, where $L^2_0(\Omega) = \{g \in L^2(\Omega) |\ \int_\Omega g \, \mathrm{d}x = 0\}$. Denote $\rho = d_2^{-1}$ and $\mathbf{P}z = z - \frac{1}{|\Omega|} \int_\Omega z \mathrm{d}x$. We make the decomposition:

$$v = \tilde{v} + \xi, \quad z = \tilde{z} + \eta,$$

where $\tilde{v}, \tilde{z} \in W^{2,2}_{\nu,0}$ and $\xi, \eta \in \mathbf{R}^+$. Define

$$\begin{aligned} F(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= (f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)^T, \\ f_1(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= \Delta u + d_1^{-1} \left[a - u(\xi + \tilde{v})^m - bu\right], \\ f_2(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= \Delta \tilde{v} + \rho \mathbf{P}\left[u(\xi + \tilde{v})^m + bu - (\xi + \tilde{v}) + c(\eta + \tilde{z} - \tilde{v} - \xi)\right], \\ f_3(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= \int_\Omega \left[u(\xi + \tilde{v})^m + bu - (\xi + \tilde{v}) + c(\eta + \tilde{z} - \tilde{v} - \xi)\right] \mathrm{d}x, \\ f_4(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= \Delta w + d_1^{-1} \left[a - w(\eta + \tilde{z})^m - bw\right], \\ f_5(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= \Delta \tilde{z} + \rho \mathbf{P}\left[w(\eta + \tilde{z})^m + bw - (\eta + \tilde{z}) + c(\xi + \tilde{v} - \tilde{z} - \eta)\right], \\ f_6(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) &= \int_\Omega \left[w(\eta + \tilde{z})^m + bw - (\eta + \tilde{z}) + c(\xi + \tilde{v} - \tilde{z} - \eta)\right] \mathrm{d}x. \end{aligned}$$

Then

$$F : \mathbf{R}^+ \times (W^{2,2}_\nu(\Omega) \times W^{2,2}_{\nu,0}(\Omega) \times \mathbf{R}^+)^2 \to (L^2(\Omega) \times L^2_0(\Omega) \times \mathbf{R})^2,$$

and, for any $\rho > 0$, (u, v, w, z) solves (1.3) is equivalent to solving

$$F(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) = 0.$$

It is clear that, for any ρ, we have $F(\rho, u^*, 0, v^*, w^*, 0, z^*) = 0$.

Let Φ be the Fréchet derivative of F at $(0, u^*, 0, v^*, w^*, 0, z^*)$ with respect to $(u, \tilde{v}, \xi, w, \tilde{z}, \eta)$. It is easy to see that

$$\Phi \equiv D_{(u, \tilde{v}, \xi, w, \tilde{z}, \eta)} F(0, u^*, 0, v^*, w^*, 0, z^*) : \\ \left(W_\nu^{2,2}(\Omega) \times W_{\nu,0}^{2,2}(\Omega) \times \mathbf{R}\right)^2 \to \left(L^2(\Omega) \times L_0^2(\Omega) \times \mathbf{R}\right)^2,$$

and

$$\Phi(\hat{u}, \hat{v}, \hat{\xi}, \hat{w}, \hat{z}, \hat{\eta}) = \begin{pmatrix} \Delta\hat{u} + d_1^{-1}\Big[-(a^m + b)\hat{u} - \frac{ma^m}{a^m+b}(\hat{v} + \hat{\xi})\Big] \\ \Delta\hat{v} \\ \int_\Omega \Big[(a^m + b)\hat{u} + \Big(\frac{ma^m}{a^m + b} - c - 1\Big)(\hat{v} + \hat{\xi}) + c(\hat{z} + \hat{\eta})\Big]\mathrm{d}x \\ \Delta\hat{w} + d_1^{-1}\Big[-(a^m + b)\hat{w} - \frac{ma^m}{a^m+b}(\hat{z} + \hat{\eta})\Big] \\ \Delta\hat{z} \\ \int_\Omega \Big[(a^m + b)\hat{w} + \Big(\frac{ma^m}{a^m + b} - c - 1\Big)(\hat{z} + \hat{\eta}) + c(\hat{v} + \hat{\xi})\Big]\mathrm{d}x \end{pmatrix}.$$

In order to use the implicit function theorem, we have to verify that Φ is both invertible and surjective. In fact, assume that $\Phi(\hat{u}, \hat{v}, \hat{\xi}, \hat{w}, \hat{z}, \hat{\eta}) = (0, 0, 0, 0, 0, 0)$, then $\hat{v} \equiv 0$ and $\hat{z} \equiv 0$. In the following, we will prove $\hat{u} = \hat{w} = \hat{\xi} = \hat{\eta} = 0$ and so Φ is invertible. In view of $\hat{v} \equiv 0$ and $\hat{z} \equiv 0$, $\Phi(\hat{u}, \hat{v}, \hat{\xi}, \hat{w}, \hat{z}, \hat{\eta}) = (0, 0, 0, 0, 0, 0)$ becomes as following:

$$\begin{cases} \Delta\hat{u} + d_1^{-1}\Big[-(a^m + b)\hat{u} - \frac{ma^m}{a^m+b}\hat{\xi}\Big] = 0 & \text{in } \Omega, \\ \int_\Omega \Big[(a^m + b)\hat{u} + \Big(\frac{ma^m}{a^m + b} - c - 1\Big)\hat{\xi} + c\hat{\eta}\Big]\mathrm{d}x = 0, & \\ \Delta\hat{w} + d_1^{-1}\Big[-(a^m + b)\hat{w} - \frac{ma^m}{a^m+b}\hat{\eta}\Big] = 0 & \text{in } \Omega, \\ \int_\Omega \Big[(a^m + b)\hat{w} + \Big(\frac{ma^m}{a^m + b} - c - 1\Big)\hat{\eta} + c\hat{\xi}\Big]\mathrm{d}x = 0, & \\ \partial_\nu\hat{u} = \partial_\nu\hat{w} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Multiplying the first equation of (4.5) with $(a^m + b)\hat{u} + \frac{ma^m}{a^m+b}\hat{\xi}$ and then integrating over Ω, noting that $\hat{\xi} \in \mathbf{R}$, we obtain

$$0 \le d_1(a^m + b)\int_\Omega |\nabla\hat{u}|^2\mathrm{d}x = -\int_\Omega \Big[(a^m + b)\hat{u} + \frac{ma^m}{a^m + b}\hat{\xi}\Big]^2\mathrm{d}x \le 0.$$

Hence

$$\hat{u} \equiv -\frac{ma^m}{(a^m+b)^2}\hat{\xi}. \tag{4.6}$$

Substituting (4.6) into the second equation of (4.5), we have

$$\int_\Omega [(-1-c)\hat{\xi} + c\hat{\eta}]\mathrm{d}x = 0.$$

In view of $\hat{\xi}, \hat{\eta} \in \mathbf{R}$, one gets

$$(-1-c)\hat{\xi} + c\hat{\eta} = 0. \tag{4.7}$$

Similarly, by the third and fourth equations of (4.5) we obtain

$$\hat{w} \equiv -\frac{ma^m}{(a^m+b)^2}\hat{\eta}, \tag{4.8}$$

and

$$(-1-c)\hat{\eta} + c\hat{\xi} = 0. \tag{4.9}$$

(4.7) combined with (4.9) yields $\hat{\xi} = \hat{\eta} = 0$. By (4.6) and (4.8) we have $\hat{u} = \hat{w} = 0$ and so Φ is invertible. Similarly, we also easily see that Φ is a surjection. By the implicit function theorem, there exists $\rho_0, r_0 > 0$ such that $(u^*, 0, v^*, w^*, 0, z^*)$ is the unique solution of

$$F(\rho, u, \tilde{v}, \xi, w, \tilde{z}, \eta) = 0 \qquad \text{in } [0, \rho_0] \times B_{r_0}(u^*, 0, v^*, w^*, 0, z^*),$$

where $B_{r_0}(u^*, 0, v^*, w^*, 0, z^*)$ denotes the open ball in $(W_\nu^{2,2}(\Omega) \times W_{\nu,0}^{2,2}(\Omega) \times \mathbf{R}^+)^2$ centered at $(u^*, 0, v^*, w^*, 0, z^*)$ with radius r_0. Taking smaller ρ_0 and r_0 if necessary, we can deduce the conclusion (i) in Theorem 4.1 by use of (i) of Lemma 4.1.

The proof of (ii) is similar. Define $F(a, u, v, w, z) : \mathbf{R}^+ \times (W_\nu^{2,2}(\Omega))^4 \to (L^2(\Omega))^4$, by

$$F(a, u, v, w, z) = \begin{pmatrix} d_1\Delta u + a - uv^m - bu \\ d_2\Delta v + uv^m + bu - v + c(z - v) \\ d_1\Delta w + a - wz^m - bw \\ d_2\Delta z + wz^m + bw - z + c(v - z) \end{pmatrix}.$$

It is easy to see that

$$
\begin{aligned}
&D_{(u,v,w,z)}F(0,0,0,0,0)\\
&=\begin{pmatrix} d_1\Delta - b & 0 & 0 & 0\\ b & d_2\Delta - c - 1 & 0 & c\\ 0 & 0 & d_1\Delta - b & 0\\ 0 & c & b & d_2\Delta - c - 1 \end{pmatrix}
\end{aligned}
$$

is a bijection from $(W_\nu^{2,2}(\Omega))^4 \to (L^2(\Omega))^4$. Thus, (ii) of Lemma 4.1 and the implicit function theorem yield our assertion. This concludes our proof. □

5 Existence of non-constant positive solutions

This section is devoted to the existence of non-constant positive solutions to (1.3). For our purpose, we start with some preliminary results.

First of all, we write

$$
\mathbf{u} = (u, v, w, z), \quad \mathbf{u}^* = (u^*, v^*, w^*, z^*),
$$

$$
\mathbf{G}(\mathbf{u}) = \begin{pmatrix} d_1^{-1}(a - uv^m - bu) \\ d_2^{-1}\left[uv^m + bu - v + c(z - v)\right] \\ d_1^{-1}(a - wz^m - bw) \\ d_2^{-1}\left[wz^m + bw - z + c(v - z)\right] \end{pmatrix}
$$

and

$$
\mathbf{A} = \begin{pmatrix} -\frac{1}{d_1}(a^m + b) & -\frac{1}{d_1}\frac{ma^m}{a^m+b} & 0 & 0 \\ \frac{1}{d_2}(a^m + b) & \frac{1}{d_2}\left(\frac{ma^m}{a^m+b} - c - 1\right) & 0 & \frac{c}{d_2} \\ 0 & 0 & -\frac{1}{d_1}(a^m + b) & -\frac{1}{d_1}\frac{ma^m}{a^m+b} \\ 0 & \frac{c}{d_2} & \frac{1}{d_2}(a^m + b) & \frac{1}{d_2}\left(\frac{ma^m}{a^m+b} - c - 1\right) \end{pmatrix}.
$$

Then $D_{\mathbf{u}}\mathbf{G}(\mathbf{u}^*) = \mathbf{A}$. Moreover, (1.3) can be written as

$$
\begin{cases} -\Delta\mathbf{u} = \mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \partial_\nu \mathbf{u} = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}
$$

and $\mathbf{u}$ is a positive solution to (5.1) if and only if

$$
\mathcal{F}(\mathbf{u}) := \mathbf{u} - (\mathbf{I} - \Delta)^{-1}(\mathbf{G}(\mathbf{u}) + \mathbf{u}) = 0 \quad \text{on } \mathbf{X},
$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ with the homogeneous Neumann boundary condition. A direct computation gives

$$
D_{\mathbf{u}}\mathcal{F}(\mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\mathbf{A} + \mathbf{I}). \tag{5.2}
$$

In order to apply the degree theory to obtain the existence of non-constant positive solutions, our first aim is to compute the index of $\mathcal{F}(\mathbf{u})$ at $\mathbf{u}^*$. By the Leray–Schauder Theorem (see [44]), we have that if 0 is not the eigenvalue of (5.2), then

$$\text{index}(\mathcal{F}, \mathbf{u}^*) = (-1)^r,$$

where r is the number of negative eigenvalues of (5.2).

A straightforward computation shows that, for each integer $j \geq 0$, $\mathbf{X}_j$ is invariant under $D_{\mathbf{u}}\mathcal{F}(\mathbf{u}^*)$, and ξ_j is an eigenvalue of $D_{\mathbf{u}}\mathcal{F}(\mathbf{u}^*)$ on $\mathbf{X}_j$ if and only if it is an eigenvalue of the matrix $\frac{1}{1+\mu_j}(\mu_j\mathbf{I} - \mathbf{A})$. Thus, $D_{\mathbf{u}}\mathcal{F}(\mathbf{u}^*)$ is invertible if and only if, for all $j \geq 0$, the matrix $\mu_j\mathbf{I} - \mathbf{A}$ is nonsingular. Denote

$$H(a, b, c, m, d_1, d_2, \mu) := \det(\mu\mathbf{I} - \mathbf{A}),$$

we also have that, if $H(a, b, c, m, d_1, d_2, \mu_j) \neq 0$, the number of negative eigenvalues of $D_{\mathbf{u}}\mathcal{F}(\mathbf{u}^*)$ on $\mathbf{X}_j$ is odd if and only if $H(a, b, c, m, d_1, d_2, \mu_j) < 0$.

Note that $m(\mu_j)$ is the algebraical multiplicity of μ_j. By similar arguments as in [18], it can be shown that the following proposition holds.

Proposition 5.1 *Suppose that, for all $j \geq 0$, the matrix $\mu_j\mathbf{I} - \mathbf{A}$ is nonsingular. Then*

$$\text{index}(\mathcal{F}(\cdot), \mathbf{u}^*) = (-1)^r, \quad where\ \ r = \sum_{j\geq 0,\ H(a,b,c,m,d_1,d_2,\mu_j)<0} m(\mu_j).$$

To compute index$(\mathcal{F}(\cdot), \mathbf{u}^*)$, we have to consider the sign of $H(a, b, c, m, d_1, d_2, \mu)$.

$$\begin{aligned}
&H(a, b, c, m, d_1, d_2, \mu)\\
&= \begin{vmatrix} \mu + \frac{1}{d_1}(a^m + b) & \frac{1}{d_1}\frac{ma^m}{a^m+b} & 0 & 0 \\ -\frac{1}{d_2}(a^m + b) & \mu - \frac{1}{d_2}\left(\frac{ma^m}{a^m+b} - c - 1\right) & 0 & -\frac{c}{d_2} \\ 0 & 0 & \mu + \frac{1}{d_1}(a^m + b) & \frac{1}{d_1}\frac{ma^m}{a^m+b} \\ 0 & -\frac{c}{d_2} & -\frac{1}{d_2}(a^m + b) & \mu - \frac{1}{d_2}\left(\frac{ma^m}{a^m+b} - c - 1\right) \end{vmatrix}\\
&= \begin{vmatrix} \mu + \frac{1}{d_1}(a^m + b) & \frac{1}{d_1}\frac{ma^m}{a^m+b} \\ -\frac{1}{d_2}(a^m + b) & \mu - \frac{1}{d_2}\left(\frac{ma^m}{a^m+b} - c - 1\right) \end{vmatrix} \cdot \begin{vmatrix} \mu + \frac{1}{d_1}(a^m + b) & \frac{1}{d_1}\frac{ma^m}{a^m+b} \\ -\frac{1}{d_2}(a^m + b) & \mu - \frac{1}{d_2}\left(\frac{ma^m}{a^m+b} - c - 1\right) \end{vmatrix}\\
&\quad + \begin{vmatrix} \mu + \frac{1}{d_1}(a^m + b) & 0 \\ -\frac{1}{d_2}(a^m + b) & -\frac{c}{d_2} \end{vmatrix} \cdot \begin{vmatrix} 0 & \mu + \frac{1}{d_1}(a^m + b) \\ -\frac{c}{d_2} & -\frac{1}{d_2}(a^m + b) \end{vmatrix}\\
&= \Big[\mu^2 + \Big(\frac{a^m + b}{d_1} - \frac{1}{d_2}\Big(\frac{ma^m}{a^m + b} - 1 - 2c\Big)\Big)\mu + \frac{1}{d_1 d_2}(1 + 2c)(a^m + b)\Big]\\
&\quad \times\Big[\mu^2 + \Big(\frac{a^m + b}{d_1} - \frac{1}{d_2}\Big(\frac{ma^m}{a^m + b} - 1\Big)\Big)\mu + \frac{1}{d_1 d_2}(a^m + b)\Big]. \qquad (5.3)
\end{aligned}$$

Assume that

$$\theta_1 := \frac{1}{d_2}\Big(\frac{ma^m}{a^m+b}-1\Big)-\frac{a^m+b}{d_1}>0, \tag{5.4}$$

$$\theta_2 := \frac{1}{d_2}\Big(\frac{ma^m}{a^m+b}-1-2c\Big)-\frac{a^m+b}{d_1}>0, \tag{5.5}$$

$$\theta_1^2-\frac{4(a^m+b)}{d_1d_2}>0, \tag{5.6}$$

and

$$\theta_2^2-\frac{4(1+2c)(a^m+b)}{d_1d_2}>0. \tag{5.7}$$

Then $H(a,b,c,m,d_1,d_2,\mu)=0$ has exactly four positive solutions $\mu_1^* < \mu_2^* < \mu_3^* < \mu_4^*$ given by

$$\mu_1^* := \mu_1^*(a,b,d_1,d_2)=\frac{1}{2}\theta_1-\frac{1}{2}\sqrt{\theta_1^2-\frac{4(a^m+b)}{d_1d_2}},$$

$$\mu_2^* := \mu_2^*(a,b,c,d_1,d_2)=\frac{1}{2}\theta_2-\frac{1}{2}\sqrt{\theta_2^2-\frac{4(a^m+b)(1+2c)}{d_1d_2}},$$

$$\mu_3^* := \mu_3^*(a,b,c,d_1,d_2)=\frac{1}{2}\theta_2+\frac{1}{2}\sqrt{\theta_2^2-\frac{4(a^m+b)(1+2c)}{d_1d_2}},$$

$$\mu_4^* := \mu_4^*(a,b,d_1,d_2)=\frac{1}{2}\theta_1+\frac{1}{2}\sqrt{\theta_1^2-\frac{4(a^m+b)}{d_1d_2}}.$$

Also

$$H(a,b,c,m,d_1,d_2,\mu)<0 \text{ if and only if } \mu\in(\mu_1^*,\mu_2^*)\cup(\mu_3^*,\mu_4^*).$$

In the following, we simplify the conditions (5.4)–(5.7). By (5.5) and (5.7), we have

$$\frac{1}{d_2}\left(\frac{ma^m}{a^m+b}-1-2c\right)-\frac{a^m+b}{d_1}>2\sqrt{\frac{(a^m+b)(2c+1)}{d_1d_2}},$$

which implies

$$\frac{1}{d_2}\frac{ma^m}{a^m+b}>\left(\sqrt{\frac{2c+1}{d_2}}+\sqrt{\frac{a^m+b}{d_1}}\right)^2,$$

i.e.,

$$\frac{ma^m}{a^m+b} > \left(\sqrt{\frac{d_2}{d_1}(a^m+b)} + \sqrt{2c+1}\right)^2. \tag{5.8}$$

As above, by (5.4) and (5.6), we have

$$\frac{ma^m}{a^m+b} > \left(\sqrt{\frac{d_2}{d_1}(a^m+b)} + 1\right)^2. \tag{5.9}$$

Obviously, (5.8) implies (5.9). On the other hand, if (5.8) holds, we can easily obtain that the four inequalities (5.4)–(5.7) hold. Therefore, the inequality (5.8) is equivalent to the system of inequalities being imposed of (5.4)–(5.7).

Remark 5.1 Let $m > 1$ and $(2c+1-m)a^m + (2c+1)b < 0$ be satisfied. Then (5.8) holds if d_2 is small enough or d_1 is large enough.

By applying similar arguments as in [7], we can claim the main result of this section as follows.

Theorem 5.1 *Let $m > 1$ be fixed. Assume that* (5.8) *holds and there exist $0 \le i < j < h < l$ such that $u_1^* \in (\mu_i, \mu_{i+1})$, $u_2^* \in (\mu_j, \mu_{j+1})$, $u_3^* \in (\mu_h, \mu_{h+1})$, $u_4^* \in (\mu_l, \mu_{l+1})$ and $\sum_{k=i+1}^{j} m(\mu_k) + \sum_{k=h+1}^{l} m(\mu_k)$ is odd. Then* (1.3) *has at least one non-constant positive solution.*

Proof By Theorem 4.1 (i) and (5.3) we can fix D large enough, such that system (1.3) with $d_2 = D$ has no non-constant solutions and

$$H(a, b, c, m, d_1, D, \mu) > 0$$

for all $\mu \ge 0$. Moreover, by Corollary 3.1 there exists $M > 0$ depending on a, b, c, m, d_1, d_2 such that for any $\hat{d} > d_2$, any positive solution (u, v, w, z) of (1.3) with diffusion coefficients d_1 and $\hat{d}$ satisfies

$$\frac{1}{M} < u, v, w, z < M \quad \text{in } \overline{\Omega}.$$

Set

$$\Theta = \left\{(u, v, w, z) \in \left[C^1(\overline{\Omega})\right]^4 \mid \frac{1}{M} < u, v, w, z < M\right\},$$

and define

$$\mathbf{\Phi} : [0, 1] \times \Theta \to \left[C^1(\overline{\Omega})\right]^4,$$

by

$$\mathbf{\Phi}(t,\mathbf{u}) = (\mathbf{I}-\Delta)^{-1}\begin{pmatrix} u+\frac{1}{d_1}(a-uv^m-bu) \\ v+\left(\frac{1-t}{D}+\frac{t}{d_2}\right)(uv^m+bu-v+c(z-v)) \\ w+\frac{1}{d_1}(a-wz^m-bw) \\ z+\left(\frac{1-t}{D}+\frac{t}{d_2}\right)(wz^m+bw-z+c(v-z)) \end{pmatrix}.$$

It is easy to see that solving (1.3) is equivalent to finding a fixed point of $\mathbf{\Phi}(1,\cdot)$ in Θ. Furthermore, from the definition of Θ and Corollary 3.1, we have that $\mathbf{\Phi}(t,\cdot)$ has no fixed points in $\partial\Theta$ for all $0 \le t \le 1$. Since $\mathbf{\Phi}(t,\cdot):[0,1]\times\Theta \to [C^1(\overline{\Omega})]^4$ is compact, the degree $\deg(\mathbf{I}-\mathbf{\Phi}(t,\cdot),\Theta,0)$ is well defined. By the homotopy invariance of degree, we can conclude

$$\deg(\mathbf{I}-\mathbf{\Phi}(1,\cdot),\Theta,0) = \deg(\mathbf{I}-\mathbf{\Phi}(0,\cdot),\Theta,0). \tag{5.10}$$

Recall that the choice of D, we have that $H(a,b,c,m,d_1,D,\mu) > 0$ and $\mathbf{u}^*$ is the only fixed point of $\mathbf{\Phi}(0,\cdot)$. From Proposition 5.1, it follows that

$$\deg(\mathbf{I}-\mathbf{\Phi}(0,\cdot),\Theta,0) = \text{index}(\mathbf{I}-\mathbf{\Phi}(0,\cdot),\mathbf{u}^*) = 1. \tag{5.11}$$

On the contrary, we assume that (1.3) has no non-constant solution. By Proposition 5.1 again, we obtain that

$$\begin{aligned}\deg(\mathbf{I}-\mathbf{\Phi}(1,\cdot),\Theta,0) &= \text{index}(\mathbf{I}-\mathbf{\Phi}(1,\cdot),\mathbf{u}^*) \\ &= (-1)^{\sum_{k=i+1}^{j} m(\mu_k)+\sum_{j=h+1}^{l} m(\mu_k)} = -1. \end{aligned} \tag{5.12}$$

Now, from (5.10)–(5.12), we get a contradiction. Therefore there exists a non-constant solution of (1.3), and the proof is complete. □

Corollary 5.1 *Let $a, b, c, d_1 > 0$ and $m > 1$ be fixed. Suppose that*

$$(2c+1-m)a^m + (2c+1)b < 0 \tag{5.13}$$

and the eigenvalue μ_j is simple for each $j \ge 1$. Then, there exists an interval sequence $\{(\lambda_n,\Lambda_n)\}_{n=1}^{\infty}$ with $\lambda_n,\ \Lambda_n \to 0$ as $n \to \infty$, such that (1.3) has at least one non-constant positive solution for all $d_2 \in (\lambda_n,\Lambda_n)$.

Proof In view of (5.13), condition (5.8) holds for small values of $d_2 > 0$. Also as $d_2 \to 0$, we have

$$\mu_1^* \to \frac{(a^m+b)^2}{d_1\,[(m-1)a^m-b]},\quad \mu_2^* \to \frac{(1+2c)(a^m+b)^2}{d_1\,[(m-1-2c)a^m-(1+2c)b]},$$

and

$$\mu_3^* \to +\infty,\quad \mu_4^* \to +\infty.$$

Moreover,

$$\mu_4^* - \mu_3^* = \frac{1}{2}(\theta_1 - \theta_2) + \frac{1}{2}\left\{\sqrt{\theta_1^2 - \frac{4(a^m+b)}{d_1 d_2}} - \sqrt{\theta_2^2 - \frac{4(1+2c)(a^m+b)}{d_1 d_2}}\right\}.$$

A direct calculation shows that $\mu_4^* - \mu_3^* \to +\infty$ as $d_2 \to 0$. Thus, we can find a sequence of intervals $\{(\lambda_n, \Lambda_n)\}_{n=1}^{\infty}$ such that

$$\sum_{k\geq 1,\ \mu_1^*<\mu_k<\mu_2^*} m(\mu_k) + \sum_{k\geq 1,\ \mu_3^*<\mu_k<\mu_4^*} m(\mu_k) \quad \text{is odd}$$

for all $d_2 \in (\lambda_n, \Lambda_n)$ $(n \geq 1)$. The conclusion follows now from Theorem 5.1. □

Corollary 5.2 *Let $a, b, c, d_2 > 0$ and $m > 1$ be fixed. Suppose that (5.13) holds and for some $h, l \geq 1$,*

(i) $\frac{(m-1-2c)a^m-(1+2c)b}{d_2(a^m+b)} \in (\mu_h, \mu_{h+1})$, $\frac{(m-1)a^m-b}{d_2(a^m+b)} \in (\mu_l, \mu_{l+1})$;
(ii) $\sum_{k=h+1}^{l} m(\mu_k)$ *is even.*

Then, there exists $D > 0$ such that system (1.3) has at least one non-constant positive solution for any $d_1 > D$.

Proof By virtue of (5.13), we take d_1 sufficiently large such that condition (5.8) holds. For fixed a, b, c, m, d_2, we have

$$\mu_1^* \to 0, \quad \mu_2^* \to 0,$$

and

$$\mu_3^* \to \frac{(m-1-2c)a^m-(1+2c)b}{d_2(a^m+b)}, \quad \mu_4^* \to \frac{(m-1)a^m-b}{d_2(a^m+b)},$$

as $d_1 \to \infty$. Thus, for $d_1 > 0$ large enough we have

$$\mu_1^*,\ \mu_2^* \in (\mu_0, \mu_1), \quad \mu_3^* \in (\mu_h, \mu_{h+1}), \quad \mu_4^* \in (\mu_l, \mu_{l+1}).$$

By Proposition 5.1 we obtain

$$\sum_{k\geq 1,\ \mu_1^*\leq\mu_k\leq\mu_2^*} m(\mu_k) = (-1)^0 = 1.$$

Therefore

$$\sum_{k\geq 1,\ \mu_1^*\leq\mu_k\leq\mu_2^*} m(\mu_k) + \sum_{k\geq 1,\ \mu_3^*\leq\mu_k\leq\mu_4^*} m(\mu_k) \ \text{is odd}$$

since $\sum_{k=h+1}^{l} m(\mu_k)$ is even. Then, the conditions in Theorem 5.1 are fulfilled and we complete the proof. □

6 Conclusions

In this paper, we analyze a two-cell coupled isothermal chemical system with arbitrary powers of autocatalysis. It is assumed that the cells are sufficiently thin so that transverse diffusion across them can be considered to be instantaneous. So we study two identical regions, divided by a semipermeable membrane which allows the passage of autocatalyst B only, with some reaction taking place in each region. Here, we consider the general case, i.e., the autocatalysis order m is any positive number and the spatial dimension N is an arbitrary positive integer.

We summarize the effects of autocatalysis order and diffusion coefficients to the pattern formation and hope to reveal some interesting phenomena of pattern formation in chemical system. First of all, from Corollary 2.1, we know that the autocatalysis order m plays an important role in generating the stationary patterns. More precisely, if $0 < m \leq 1$ then no stationary patterns occur, while if $m > 1$, from Theorem 5.1, there exist such patterns on the condition that (5.8) holds. Secondly, from Theorem 4.1, the large diffusion rate of autocatalyst B can lead to the non-existence of spatial pattern, while from Corollaries 5.1 and 5.2, a large diffusion rate of reactant A or small diffusion rate of autocatalyst B will help the generation of patterns. These demonstrate that, in a chemical model, different autocatalysis orders or diffusions may play essentially different roles in developing spatial patterns.

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